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On the Design of Robust Static Output Feedback Controllers via Robust Stabilizability Functions

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Abstract—A key problem in control systems consists of designing robust stabilizing controllers for systems with parametric uncertainties, in particular output feedback controllers that, without requiring to measure the uncertainty, ensure robust stability of the closed-loop system. This paper aims to establish the existence of such a controller and, more generally, to determine a robust stabilizing controller that minimize a chosen cost. This problem is considered in this paper for systems whose coefficients are polynomial functions of an uncertain vector constrained into a semialgebraic set. The admissible controllers are those in a given hyper-rectangle for which the closed-loop system is well-posed. First, the class of robust stabilizability functions is introduced, i.e. the class of functions that, when evaluated for an admissible controller, are positive if and only if the controllers robustly stabilizes the system. Second, the approximation of a robust stabilizability function with a controller-dependent lower bound is proposed through a convex program exploiting a technique developed in the estimation of the robust domain of attraction. Third, the derivation of a robust stabilizing controller from the found controller-dependent lower bound is addressed through a second convex program that provides an upper bound of the optimal cost.

Index Terms—Uncertain systems, Robust control, LMI.

I. INTRODUCTION

It is well-known that systems with uncertainty is a fundamental area of control systems, see e.g. [1], [8], [16]. In fact, the mathematical model of a real dynamical system is never exactly known due to the presence of uncertain parameters that affect its coefficients. Numerous approaches have been derived for robust stability analysis of systems affected by parametric uncertainties. These approaches can be classified into different categories, mainly depending on characteristic of the system and on the tools exploited to derive the conditions for ensuring robust stability. Many of these approaches provide sufficient (and, sometimes, necessary) conditions based on Lyapunov functions that require the solution of convex optimization problems with linear matrix inequalities (LMIs). See e.g. [2], [4], [6], [9]–[11], [13]–[15] and references therein.

Unfortunately, these conditions unavoidably lead to non-convex optimization whenever applied to robust control design. In fact, whenever a controller to be designed is present, the LMIs generally become bilinear matrix inequalities (BMIs) in the unknown Lyapunov function and controller. Contrary to LMIs, BMIs can have nonconvex feasible sets, which means that one does not know how finding a solution whenever it exists. Clearly, one can fix

one of the variables (also alternatively) to change the BMIs into LMIs, but the obtained condition is sufficient only. In order to cope with this issue, some approaches have been proposed without the use of Lyapunov functions, see e.g. [7] which provides conditions for robust stability, and [12] which estimates robust stability regions.

This paper addresses the design of robust stabilizing controllers for systems with parametric uncertainties, in particular output feedback controllers that, without requiring to measure the uncertainty, ensure robust stability of the closed-loop system. The aim is to establish the existence of such a controller and, more generally, to determine a robust stabilizing controller that minimize a chosen cost. This problem is considered in this paper for systems whose coefficients are polynomial functions of an uncertain vector constrained into a semialgebraic set. The admissible controllers are those in a given hyper-rectangle for which the closed-loop system is well-posed. First, the class of robust stabilizability functions is introduced, i.e. the class of functions that, when evaluated for an admissible controller, are positive if and only if the controllers robustly stabilizes the system. Second, the approximation of a robust stabilizability function with a controller-dependent lower bound is proposed through a convex program exploiting a technique developed in the estimation of the robust domain of attraction. Third, the derivation of a robust stabilizing controller from the found controller-dependent lower bound is addressed through a second convex program that provides an upper bound of the optimal cost. The proofs of the proposed results are omitted due to lack of space, the interested readers are welcome to contact the author for details.

II. PRELIMINARIES

Notation: \mathbb{R} : real number sets; 0_n : origin of \mathbb{R}^n ; I_n : $n \times n$ identity matrix; A' : transpose of A ; $\det(A)$: determinant of matrix A ; $\text{adj}(A)$: adjoint of matrix A ; $\text{spec}(A)$: set of eigenvalues of matrix A ; $\text{col}(A)$: column vector stacking the columns of matrix A ; $A > 0$, $A \geq 0$: symmetric positive definite and symmetric positive semidefinite matrix A ; Hurwitz matrix: a matrix with all eigenvalues having negative real part; $\deg(a)$: degree of polynomial $a(x)$.

Let us consider the uncertain system

$$\begin{cases} \dot{x}(t) &= A(p)x(t) + B(p)u(t) \\ y(t) &= C(p)x(t) + D(p)u(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, $p \in \mathbb{R}^q$, and $A(p)$, $B(p)$, $C(p)$ and $D(p)$ are matrix polynomials. It is

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supposed that

$$p \in \mathcal{P} \quad (2)$$

where

$$\mathcal{P} = \{p \in \mathbb{R}^q : a_i(p) \geq 0, b_j(p) = 0, \forall i = 1, \dots, n_a, j = 1, \dots, n_b\} \quad (3)$$

and $a_i(p)$ and $b_j(p)$, $i = 1, \dots, n_a$ and $j = 1, \dots, n_b$, are polynomials. This system is controlled in closed-loop via

$$u(t) = Ky(t) \quad (4)$$

where $K \in \mathbb{R}^{n_u \times n_y}$ is the controller to be determined. It is supposed that

$$K \in \mathcal{K} \quad (5)$$

where $\mathcal{K} \subset \mathbb{R}^{n_u \times n_y}$ is the hyper-rectangle

$$\mathcal{K} = \{K \in \mathbb{R}^{n_u \times n_y} : k_{ij}^- \leq k_{ij} \leq k_{ij}^+, i = 1, \dots, n_u, j = 1, \dots, n_y\} \quad (6)$$

where $k_{ij} \in \mathbb{R}$ is the (i, j) -th entry of K , and k_{ij}^-, k_{ij}^+ are its lower and upper bounds. The closed-loop system (1)–(6) can be described as

$$\begin{cases} \dot{x}(t) = A_{cl}(K, p)x(t) \\ K \in \mathcal{K} \cap \mathcal{K}_{wp} \\ p \in \mathcal{P} \end{cases} \quad (7)$$

where

$$A_{cl}(K, p) = A(p) + B(p)K(I - D(p)K)^{-1}C(p) \quad (8)$$

and \mathcal{K}_{wp} is the set of controllers such that $A_{cl}(K, p)$ is well-posed. In particular, we say that $A_{cl}(K, p)$ is well-posed if

$$|\det(I - D(p)K)| \geq \rho_{wp} \quad \forall p \in \mathcal{P}, \quad (9)$$

where $\rho_{wp} > 0$ is an arbitrary small chosen threshold. Hence,

$$\mathcal{K}_{wp} = \{K \in \mathbb{R}^{n_u \times n_y} : (9) \text{ holds}\}. \quad (10)$$

The system (7) is said robustly stable if

$$\Re(\lambda) < -\rho_s \quad \forall \lambda \in \text{spec}(A_{cl}(K, p)) \quad \forall p \in \mathcal{P} \quad (11)$$

for some $\rho_s \geq 0$.

Problem. The problem is to establish the existence of a robust stabilizing controller for the system (7), i.e. the non-emptiness of the set

$$\mathcal{K}_s = \{K \in \mathcal{K} \cap \mathcal{K}_{wp} : (11) \text{ holds}\}. \quad (12)$$

In addition, we aim to determine a controller in \mathcal{K}_s that minimizes a given polynomial cost, i.e. solve

$$r^* = \inf_{K \in \mathcal{K}_s} r(K) \quad (13)$$

where $r(K)$ is a polynomial.

III. ROBUST STABILIZABILITY FUNCTIONS

The first part of the proposed method consists of building a robust stabilizability function for the system (7). Specifically, we say that $s : \mathbb{R}^{n_u \times n_y} \rightarrow \mathbb{R}$ is a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{K}_{wp}$ for the system (7) if and only if

$$K \in \mathcal{K} \cap \mathcal{K}_{wp} \Rightarrow \begin{cases} s(K) > 0 & \text{if } K \in \mathcal{K}_s \\ s(K) \leq 0 & \text{otherwise.} \end{cases} \quad (14)$$

This means that a controller in the set $\mathcal{K} \cap \mathcal{K}_{wp}$ is a robust stabilizing controller for the system (7) if and only if

$$s(K) > 0. \quad (15)$$

Here we show how to build a robust stabilizability function for the system (7) that will be exploited to solve the robust control design problem.

Let us start by defining the set

$$\mathcal{T} = \begin{cases} \{0\} & \text{if } D(p) = 0 \\ \{0, 1\} & \text{otherwise} \end{cases} \quad (16)$$

and the partition of \mathcal{K}_{wp} given by

$$\mathcal{K}_{wp} = \bigcup_{\tau \in \mathcal{T}} \mathcal{U} \quad (17)$$

where

$$\mathcal{U} = \{K \in \mathbb{R}^{n_u \times n_y} : (-1)^\tau \det(I - D(p)K) \geq \rho_{wp}, \forall p \in \mathcal{P}\}. \quad (18)$$

Next, let us consider a generic monic polynomial of degree n in a variable $\lambda \in \mathbb{C}$ expressed as

$$v(\lambda) = \lambda^n + \sum_{i=0}^{n-1} d_i \lambda^i \quad (19)$$

for some generic $d_i \in \mathbb{R}$, $i = 0, \dots, n-1$. Let us write the Routh-Hurwitz table of $v(\lambda)$ as

$$\begin{array}{ccc} e_{00} & e_{01} & \dots \\ e_{10} & e_{11} & \dots \\ \vdots & & \end{array} \quad (20)$$

where the first two rows are given by

$$\begin{aligned} e_{00} &= 1, \quad e_{01} = d_{n-2}, \dots \\ e_{10} &= d_{n-1}, \quad e_{11} = d_{n-3}, \dots \end{aligned} \quad (21)$$

while the remaining ones (for $i = 2, \dots, n$) follow the rule

$$e_{ij} = \frac{e_{i-1,0}e_{i-2,j+1} - e_{i-2,0}e_{i-1,j+1}}{e_{i-1,0}}. \quad (22)$$

We have that e_{ij} can be expressed as

$$e_{ij} = \frac{\bar{e}_{ij}}{\hat{e}_{ij}} \quad (23)$$

where \bar{e}_{ij} and \hat{e}_{ij} are polynomials. In particular, \hat{e}_{ij} is given by

$$\hat{e}_{ij} = \prod_{l=i-1, i-3, \dots} \bar{e}_{lj}. \quad (24)$$

Let $z \in \mathbb{R}^{n_z}$ be the variable defined as

$$z = \begin{pmatrix} \text{col}(K) \\ p \end{pmatrix}, \quad n_z = n_u n_y + q. \quad (25)$$

Let us express $A_{cl}(K, p)$ as

$$A_{cl}(K, p) = \frac{\bar{A}_{cl}(z)}{\det(I - D(p)K)} \quad (26)$$

where $\bar{A}_{cl}(z)$ is the matrix polynomial

$$\bar{A}_{cl}(z) = \det(I - D(p)K)A(p) + B(p)K \text{adj}(I - D(p)K)C(p) \quad (27)$$

where $\text{adj}(I - D(p)K)$ is the adjoint matrix of $I - D(p)K$. Let us define

$$W(z) = (-1)^\tau (\bar{A}_{cl}(z) + \rho_s \det(I - D(p)K)I) \quad (28)$$

and its characteristic polynomial

$$v(\lambda, z) = \det(\lambda I - W(z)). \quad (29)$$

Let $\bar{e}_{ij}(z)$ and $\hat{e}_{ij}(z)$ be the numerator and denominator in (23) obtained for $v(\lambda)$ replaced by $v(\lambda, z)$. Let us observe that $\bar{e}_{ij}(z)$ and $\hat{e}_{ij}(z)$ do exist and are polynomials for all i, j in the table (20). Let us define the set

$$\mathcal{N} = \{i = 0, \dots, n : \bar{e}_{i0}(z) \text{ is a non-positive constant}\} \quad (30)$$

and let $f_m(z)$, $m = 1, \dots, n_f$, be the non-constant polynomials among $\bar{e}_{i0}(z)$, $i = 0, \dots, n$.

The following theorem suggests a strategy to construct a robust stabilizability function for the system (7).

Theorem 1: Let $\tau \in \mathcal{T}$. If $\mathcal{N} \neq \emptyset$, then

$$(11) \text{ does not hold for any } K \in \mathcal{U}. \quad (31)$$

Hence, suppose that $\mathcal{N} = \emptyset$, and let us define

$$s(K) = \inf_{\substack{p \in \mathcal{P} \\ m=1, \dots, n_f}} f_m(z). \quad (32)$$

Then,

$$\begin{cases} s(K) > 0 \\ K \in \mathcal{K} \cap \mathcal{U} \end{cases} \Rightarrow K \in \mathcal{K}_s. \quad (33)$$

Moreover, if \mathcal{P} is compact, this condition holds in both directions, i.e. $s(K)$ is a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{U}$ for the system (7).

Theorem 1 states that a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{U}$ for the system (7) can be built using the polynomials $f_m(z)$ whenever \mathcal{P} is compact. Such a function does not exist whenever the set \mathcal{N} is nonempty because, according to (31), there cannot exist any robust stabilizing controller over the set \mathcal{U} for the system (7) in such a case. If \mathcal{P} is not compact, one can still use the function $s(K)$ to obtain a sufficient condition for identifying robust stabilizing controllers in $\mathcal{K} \cap \mathcal{U}$ according to (33).

IV. CONTROLLER-DEPENDENT LOWER BOUND

The second part of the proposed method consists of determining a controller-dependent lower bound of the robust stabilizability function $s(K)$ in (32). Let us start by expressing the set $\mathcal{K} \cap \mathcal{U}$ as

$$\mathcal{K} \cap \mathcal{U} = \{K \in \mathbb{R}^{n_u \times n_y} : c_l(z) \geq 0 \quad \forall p \in \mathcal{P}, l = 1, \dots, n_c\} \quad (34)$$

where $c_l(z)$, $l = 1, \dots, n_c$, are polynomials defined as follows. First, in order to impose $K \in \mathcal{K}$, let us define polynomials $c_l(z)$ as

$$c_l(z) = (k_{ij} - k_{ij}^-)(k_{ij}^+ - k_{ij}) \quad (35)$$

for all possible i, j . Second, in order to impose $K \in \mathcal{U}$, let us define one of the polynomials $c_l(z)$ as

$$c_l(z) = (-1)^\tau \det(I - D(p)K) - \rho_{wp}. \quad (36)$$

Let us denote with $\xi(K)$ the sought controller-dependent lower bound of the robust stabilizability function $s(K)$. We consider the case where $\xi(K)$ is a polynomial, and denote with d_ξ its degree. The idea for computing $\xi(K)$ is to exploit the technique we introduced in [3] for estimating the parameter-dependent lower bound of estimates of the robust domain of attraction, where the gap between the original function and the polynomial lower bound is minimized by maximizing the integral of the latter.

Specifically, let us define $n_g = n_f$, and, for $m = 1, \dots, n_g$, let us introduce the polynomials

$$g_m(z) = f_m(z) - \xi(K) - \sum_{i=1}^{n_a} a_i(p) \alpha_{mi}(z) - \sum_{j=1}^{n_b} b_j(p) \beta_{mj}(z) - \sum_{l=1}^{n_c} c_l(z) \gamma_{ml}(z) \quad (37)$$

where $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$, $i = 1, \dots, n_a$, $j = 1, \dots, n_b$ and $l = 1, \dots, n_c$, are polynomials to be determined. Then, let us define the function $h(\xi)$ as the function that satisfies the equation

$$h(\xi) = \int_{K \in \mathcal{K}} \psi(K) \xi(K) dK \quad (38)$$

where $\psi(K)$ is a positive polynomial over \mathcal{K} (we will discuss the choice of $\psi(K)$ in the next section). Let us observe that $h(\xi)$ is a linear function on the coefficients of $\xi(K)$ and can be easily computed since \mathcal{K} is a hyperrectangle. In particular, the coefficient of $h(\xi)$ that multiplies the coefficient of the monomial $k_{11}^{l_{11}} k_{12}^{l_{12}} \dots$ in $\psi(K) \xi(K)$ is

$$\prod_{ij} \frac{(k_{ij}^+)^{l_{ij}+1} - (k_{ij}^-)^{l_{ij}+1}}{l_{ij} + 1}. \quad (39)$$

The following theorem provides a strategy for constructing a controller-dependent lower bound of the robust stabilizability function $s(K)$ through a convex program with polynomials that are sums of squares of polynomials (SOS).

Theorem 2: Let $\tau \in \mathcal{T}$, and suppose without loss of generality that $\mathcal{N} \neq \emptyset$. For any chosen degrees of the

polynomials $\xi(K)$, $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$ let us define the convex program

$$\begin{aligned} h^* &= \sup_{\xi, \alpha_{mi}, \beta_{mj}, \gamma_{ml}} h(\xi) \\ \text{s.t. } &\begin{cases} g_m(z) \text{ is SOS} \\ \alpha_{mi}(z) \text{ is SOS} \\ \gamma_{ml}(z) \text{ is SOS} \end{cases} \quad \forall \begin{cases} m = 1, \dots, n_g \\ i = 1, \dots, n_a \\ l = 1, \dots, n_c. \end{cases} \end{aligned} \quad (40)$$

Let $\xi^*(K)$ be the polynomial $\xi(K)$ corresponding to the optimal values of the variables in (40). Then,

$$\xi^*(K) \leq s(K) \quad \forall K \in \mathcal{K} \cap \mathcal{U} \quad (41)$$

where $s(K)$ is the robust stabilizability function in (32). Therefore,

$$\xi^*(K) > 0 \text{ and } K \in \mathcal{K} \cap \mathcal{U} \Rightarrow K \in \mathcal{K}_s. \quad (42)$$

Theorem 2 states that the polynomial $\xi^*(K)$ provided by (40) is a guaranteed lower bound of the robust stabilizability function $\bar{s}(K)$ over $\mathcal{K} \cap \mathcal{U}$. This implies that, if there exists K in the set $\mathcal{K} \cap \mathcal{U}$ such that $\xi^*(K) > 0$, then K is a robust stabilizing controller for the system (7), i.e. $K \in \mathcal{K}_s$. The next section will discuss the determination of such a controller. Let us observe that \mathcal{P} is not required to be a compact set in Theorem 2. Let us also observe that there is no loss of generality in supposing that $\mathcal{N} \neq \emptyset$ because the opposite implies from Theorem 1 that there cannot exist any robust stabilizing controller in the set \mathcal{U} for the system (7). Since establishing whether a polynomial depending linearly on some unknowns is SOS amounts to checking feasibility of an LMI, it directly follows that (40) is a convex program, see e.g. [5] and references therein for details about SOS polynomials.

Let us observe that the controller-dependent lower bound $\xi^*(K)$ is obtained in Theorem 2 by maximizing the integral of $\xi(K)$ over \mathcal{K} , i.e. $h(\xi)$. This is done in order to minimize the gap between $s(K)$ and $\xi(K)$, and exploits the idea we introduced in [3] for estimating the parameter-dependent lower bound of estimates of the robust domain of attraction.

Let us also observe that the degrees of the polynomials $\xi(K)$, $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$ can be freely chosen in Theorem 2. A simple way of choosing such degrees is to choose the degree of $\xi(K)$, i.e. d_ξ , and define the minimum degrees of the polynomials $g_m(z)$ as

$$\begin{aligned} d_{g_m} &= \max\{\deg(f_m), d_\xi, \deg(a_i), \deg(b_j), \deg(c_l) \\ &\quad i = 1, \dots, n_a, j = 1, \dots, n_b, l = 1, \dots, n_c\}. \end{aligned} \quad (43)$$

Then, the degrees of the polynomials $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$ are chosen as the largest degrees such that the degree of $g_m(z)$ is even and not greater than d_{g_m} . Clearly, the degrees of $\alpha_{mi}(z)$ and $\gamma_{ml}(z)$ must be even since these polynomials are required to be SOS in Theorem 2. This choice will be adopted in the remaining part of the paper.

V. ROBUST STABILIZING CONTROLLER

The third part of the proposed method consists of determining a robust stabilizing controller that minimizes the cost $r(K)$ from the found controller-dependent lower bound $\xi^*(K)$. The idea is to minimize $r(K)$ over the set of admissible controllers for which $\xi^*(K)$ is positive, i.e.

$$\begin{aligned} \hat{r}^* &= \inf_K r(K) \\ \text{s.t. } &\begin{cases} \xi^*(K) \geq \varepsilon \\ K \in \mathcal{K} \cap \mathcal{U} \end{cases} \end{aligned} \quad (44)$$

where $\varepsilon > 0$ is an arbitrary small constant introduced for considering positive values only of $\xi^*(K)$. This suggests that the free polynomial $\psi(K)$ in (38) can be chosen in order to facilitate the increase of $\xi(K)$ whenever $r(K)$ is small. This can be simply achieved with

$$\psi(K) = \psi_0 - r(K) \quad (45)$$

where $\psi_0 > 0$ is such that $\psi(K)$ is positive over \mathcal{K} .

In order to solve (44), let us define the polynomial

$$\begin{aligned} w(z) &= r(K) - \theta - (\xi^*(K) - \varepsilon)\delta(z) - \sum_{i=1}^{n_a} a_i(p)\alpha_i(z) \\ &\quad - \sum_{j=1}^{n_b} b_j(p)\beta_j(z) - \sum_{l=1}^{n_c} c_l(z)\gamma_l(z) \end{aligned} \quad (46)$$

where $\theta \in \mathbb{R}$ is a scalar to be determined, and $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$, $i = 1, \dots, n_a$, $j = 1, \dots, n_b$ and $l = 1, \dots, n_c$, are polynomials to be determined. For any degrees of these polynomials, let us define the convex program

$$\begin{aligned} \theta^* &= \sup_{\theta, \alpha_i, \beta_j, \gamma_l, \delta} \theta \\ \text{s.t. } &\begin{cases} w(z) \text{ is SOS} \\ \alpha_i(z) \text{ is SOS } \forall i = 1, \dots, n_a \\ \gamma_l(z) \text{ is SOS } \forall l = 1, \dots, n_c \\ \delta(z) \text{ is SOS.} \end{cases} \end{aligned} \quad (47)$$

Using arguments similar to those of the proof of Theorem 2, it is easy to see that θ^* is a lower bound of \hat{r}^* , i.e.

$$\theta^* \leq \hat{r}^*. \quad (48)$$

The degrees of the polynomials $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$ in (47) can be freely chosen, and one can adopt a rule similar to the one provided for choosing the degrees of the polynomials in Theorem 2. Indeed, let us define the minimum degree of the polynomial $w(K)$ as

$$\begin{aligned} d_w &= \max\{\deg(r), d_\xi, \deg(a_i), \deg(b_j), \deg(c_l) \\ &\quad i = 1, \dots, n_a, j = 1, \dots, n_b, l = 1, \dots, n_c\}. \end{aligned} \quad (49)$$

Then, the degrees of the polynomials $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$ are chosen as the largest degrees such that the degree of $w(z)$ is even and not greater than $d_w + \nu$, where ν is a free nonnegative integer. Clearly, the degrees of $\alpha_i(z)$, $\gamma_l(z)$ and $\delta(z)$ must be even since these polynomials are required to be SOS in Theorem 2. The numerical examples in Section VI are solved with the simple choice $\nu = 0$.

The following result provides a strategy for determining a robust stabilizing controller that minimizes the cost $r(K)$ from the solution of (47).

Theorem 3: Let $w^*(z)$ be the polynomial $w(z)$ corresponding to the optimal values of the variables in (47). The lower bound θ^* of \hat{r}^* satisfies

$$\theta^* = \hat{r}^* \quad (50)$$

if and only if there exist K and p such that

$$\begin{cases} w^*(z) = 0 \\ r(K) = \theta^* \\ \xi^*(K) \geq \varepsilon \\ K \in \mathcal{K} \cap \mathcal{U} \\ p \in \mathcal{P}. \end{cases} \quad (51)$$

In such a case, K is a robust stabilizing controller for the system (7), i.e. $K \in \mathcal{K}_s$. Moreover, θ^* is an upper bound of the optimal cost r^* , i.e.

$$\theta^* \geq r^*. \quad (52)$$

Theorem 3 suggests that one can obtain a robust stabilizing controller for the system (7) by looking for K and p such that the condition (51) holds. If there exist such K and p , θ^* coincides with \hat{r}^* and is an upper bound of r^* . Moreover, such a K is a robust stabilizing controller for the system (7) if $\xi^*(K) > 0$. If there does not exist any K and p satisfying the condition (51), one should repeat the computation of $w^*(z)$ with a larger value of ν as it will become clear in the sequel.

In order to look for K and p such that the condition (51) holds, let us define the set

$$\mathcal{C} = \{z : w^*(z) = 0\}. \quad (53)$$

It turns out that, in non-degenerate cases, \mathcal{C} is a finite set, which can be found through linear algebra operations since $w^*(z)$ is SOS. Once \mathcal{C} has been found, one checks whether any z in the set \mathcal{C} satisfies the condition (51) holds.

We conclude the paper claiming that, under some mild conditions, the proposed methodology is not only sufficient but also necessary for determining a robust stabilizing controller, and the upper bound θ^* asymptotically converges to r^* .

VI. EXAMPLES

A. Example 1

Let us start by considering the second-order uncertain system with a scalar controller described by

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} -1-p & 2p \\ -2 & -3+2p \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} -1 & 1 \end{pmatrix} x(t) - 0.2u(t) \\ p \in \mathcal{P} = [-1, 1]. \end{cases}$$

The problem is to find a controller K in

$$\mathcal{K} = [-5, 5]$$

such that $u(t) = Ky(t)$ ensures robust stability according to (11) with $\rho_s = 0.1$ and minimizes the cost

$$r(K) = K^2.$$

In order to ensure that the closed-loop system is well-posed, we consider (9) with $\rho_{wp} = 0.1$.

First of all, let us observe that the autonomous system is not robustly stable since for $K = 0$ and $p = -1$ one has

$$\text{spec}(A_{cl}(K, p)) = \{-5.702, 0.702\}.$$

Since $D(p)$ is present, it follows from (16)–(18) that $\mathcal{T} = \{0, 1\}$ and

$$\mathcal{U} = \begin{cases} [-4.5, 5] & \text{if } \tau = 0 \\ \emptyset & \text{if } \tau = 1. \end{cases}$$

Since $\mathcal{U} = \emptyset$ for $\tau = 1$, we consider only $\tau = 0$.

We describe \mathcal{P} as in (3) by defining $n_a = 1$, $n_b = 0$ and $a(p) = 1 - p^2$. From the table (20) we find that $\mathcal{N} = \emptyset$ and, hence, we proceed to computing the controller-dependent lower bound $\xi^*(K)$ with (40).

For $d_\xi = 2$ we find $h^* = -50.873$ and

$$\xi^*(K) = -4.591 - 3.556K - 0.608K^2.$$

Hence, we solve (47), which provides

$$\theta^* = 3.699.$$

Next, let us Theorem 3. We find that (51) holds with

$$K = -1.923.$$

Therefore, K is a robust stabilizing controller, and θ^* is an upper bound of the optimal cost r^* .

Repeating for $d_\xi = 3$ we find $h^* = -50.432$ and

$$\xi^*(K) = -4.510 - 3.636K - 0.611K^2 + 0.008K^3.$$

Hence, we solve (47), which provides

$$\theta^* = 3.220.$$

Next, let us Theorem 3. We find that (51) holds with

$$K = -1.794.$$

Figure 1 illustrates the found results for $d_\xi = 3$.

B. Example 2

Let us consider the third-order uncertain system with a bivariate controller described by

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 1+p \\ -p & 0 & 1 \\ -2-2p & -6 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 2+p \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 2-p & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) \\ p \in \mathcal{P} = [-1, 1]. \end{cases}$$

The problem is to find a controller $K = (k_1, k_2)$ in

$$\mathcal{K} = [-5, 5]^2.$$

such that $u(t) = Ky(t)$ ensures robust stability according to (11) with $\rho_s = 0.01$ and minimizes the cost

$$r(K) = k_1^2 + k_2^2.$$

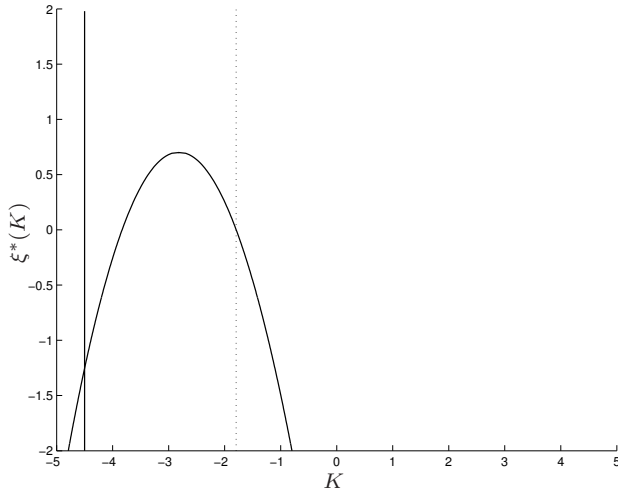


Fig. 1. Example 1: controller-dependent lower bound $\xi^*(K)$ (solid curve), robust stabilizing controller K (dotted), and boundary of \mathcal{U} (solid line) for $d_\xi = 3$.

Since $D(p)$ is absent, the system is well-posed for all K and (9) is not needed. Hence, we consider only $\tau = 0$.

First of all, let us observe that the autonomous system is not robustly stable since for $K = (0, 0)$ and $p = -1$ one has

$$\text{spec}(A_{cl}(K, p)) = \{-1.728 \pm 1.895\sqrt{-1}, 0.456\}.$$

For $d_\xi = 3$ we find $h^* = -2549.582$ and

$$\begin{aligned} \xi^*(K) = & -6.512 - 3.546k_1 + 3.626k_2 - 0.709k_1^2 \\ & - 0.690k_1k_2 - 1.183k_2^2 + 0.026k_1^3 \\ & - 0.188k_1^2k_2 - 0.141k_1k_2^2 - 0.104k_2^3. \end{aligned}$$

Hence, we solve (47), which provides

$$\theta^* = 2.746.$$

Next, let us Theorem 3. We find that (51) holds with

$$K = (-1.267, 1.068).$$

Therefore, K is a robust stabilizing controller, and θ^* is an upper bound of the optimal cost r^* . Figure 2 illustrates the found results.

VII. CONCLUSIONS

This paper has addressed the design of robust stabilizing controllers for systems with parametric uncertainties. The proposed method is based on the introduction of the class of robust stabilizability functions, on the approximation of one of these functions through a polynomial lower bound, and on the derivation of a robust stabilizing controller candidate that minimizes a given cost. The proposed method exploits convex optimization with SOS polynomials and has the advantage of not introducing BMIs while being asymptotically nonconservative under some mild conditions.

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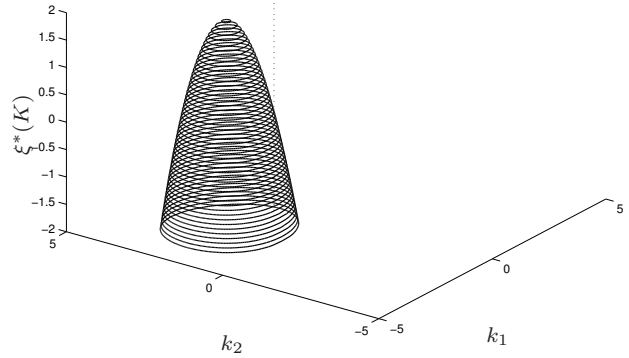


Fig. 2. Example 2: controller-dependent lower bound $\xi^*(K)$ (solid) and robust stabilizing controller K (dotted).

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